Random matrix analysis of complex networks

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We study complex networks under random matrix theory (RMT) framework. Using nearest-neighbor and next-nearest-neighbor spacing distributions we analyze the eigenvalues of the adjacency matrix of various model networks, namely, random, scale-free, and small-world networks. These distributions follow the Gaussian orthogonal ensemble statistic of RMT. To probe long-range correlations in the eigenvalues we study spectral rigidity via the Δ_3 statistic of RMT as well. It follows RMT prediction of linear behavior in semi-logarithmic scale with the slope being $\sim 1/\pi^2$. Random and scale-free networks follow RMT prediction for very large scale. A small-world network follows it for sufficiently large scale, but much less than the random and scale-free networks.

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I. INTRODUCTION

Random matrix theory (RMT), initially proposed to explain statistical properties of nuclear spectra, had successful predictions for the spectral properties of different complex systems such as disordered systems, quantum chaotic systems, large complex atoms, etc., followed by numerical and experimental verifications in the last few decades [1,2]. Quantum graphs, which model the systems of interest in quantum chemistry, solid state physics, and transmission of waves, have also been studied under the RMT framework [3]. Recently, RMT has been shown to be useful also in understanding the statistical properties of empirical cross-correlation matrices appearing in the study of multivariate time series of the following: price fluctuations in the stock market [4,5], EEG data of the brain [6], variation of different atmospheric parameters [7], etc.

In our previous studies [8,9] complex networks have been analyzed under the RMT framework. These works consider nearest-neighbor spacing distribution (NNSD) of eigenvalues spectra of adjacency and Laplacian matrices of various extensively studied networks. The NNSD gives the probability for finding neighboring eigenvalues with a given spacing, and it follows two universal properties depending upon underlying correlations among the eigenvalues. For the correlated eigenvalues, NNSD follows Gaussian orthogonal ensemble (GOE) statistics of RMT, whereas it follows Poissonian statistics for the uncorrelated eigenvalues. One of the main advantages of the RMT approach is that depending on the nature of eigenvalues correlations one can separate the system dependent part from the random universal part, which are intermingled due to the complexity of the system [2,4–7]. RMT analysis for the various networks shows that the NNSD of complex networks also follows universal GOE statistics of RMT [8]. This finding suggests that different results of GOE statistics, which have successfully been applied to understand the systems coming from various fields starting from nuclei to the stock market can be applied to study networks as well. Our earlier works concentrate on the NNSD studies of networks. NNSD carries information for the correlation between two adjacent eigenvalues, but does not tell about the correlation between two far-off eigenvalues. Therefore even though NNSD follows GOE statistics of RMT, other properties may show deviations, which suggests that one cannot rely on NNSD results exclusively. To probe for long-range correlations as well, current work considers the spectral rigidity test via the well known Δ_3 statistic of RMT. It is found that the spectral rigidity of the complex networks follows RMT prediction, with scale depending upon the properties of the networks. Present work also analyzes the *next*-nearest-neighbor spacing distribution (NNNSD) of the adjacency matrix of the networks.

The paper is organized as follows: following this introduction, Sec. II explains various aspects of complex networks studies. Section III describes some basics of RMT relevant to our studies. Section IV illustrates the RMT analysis for various model networks, namely; random, scale-free, and small world. The NNSD is the most widely studied property in random matrix literature, therefore this section includes NNSD results for the above mentioned model networks [8], and presents results for NNNSD and the Δ_3 statistic of these networks. Finally, Sec. V discusses and summarizes results with some possible future directions.

II. COMPLEX NETWORKS

The last ten years have witnessed a rapid advancement in the studies of complex networks. The main concept of the network theory is to define complex systems in terms of networks of many interacting units. A few examples of such systems are interacting molecules in living cells, nerve cells in the brain, computers in Internet communication, social networks of interacting people, airport networks with flight connections, etc. [10–12]. In the graph theoretical terminology, units are called nodes and interactions are called edges [13]. Various model networks have been introduced to study the behavior of complex systems having underlying network structures. These model networks are based on simple principles, still they capture essential features of the underlying systems.

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A. Structural properties

In the random graph model of Erdös and Rényi any two nodes are randomly connected with probability p [14]. This model assumes that interactions between nodes are random. Recently, with the availability of large maps of real world networks, it has been observed that the random graph model is not appropriate for studying the behavior of real world networks. Hence many new models have been introduced. Watts and Strogatz proposed a model, popularly known as the "small-world network," which has properties of small diameter and high clustering [15]. Moreover, this model network is very sparse: a network with a very few number of edges, another property shown by many real-world networks. In addition to the above mentioned properties, Barabási and Albert show that degree distributions of many real-world networks have a power law. This implies that some nodes are much more connected than the others [16]. Barabási-Albert's scale-free model and Watts-Strogatz's small-world model have contributed immensely in understanding evolution and behavior of the real systems having network structures. Following these two new models came an outbreak in the field of networks. These studies show that real world networks have coexistence of *randomness* and *regularity* [10,17,18].

B. Spectral properties

Apart from the above mentioned investigations which focus on direct measurements of the structural properties of networks, there have been lot of studies demonstrating that properties of networks or graphs could be well characterized by the spectrum of associated adjacency matrix A [19]. For an unweighted graph, it is defined in the following way: $A_{ij}=1$, if i and j nodes are connected, and *zero* otherwise. For an undirected network, this matrix is symmetric and consequently has real eigenvalues. Eigenvalues give information about some basic topological properties of the underlying network [19,20]. Spectral properties of networks have also been used to understand some of the dynamical properties of interacting chaotic units on networks, for example, the largest eigenvalue of the adjacency matrix determines the transition to the synchronized state [21]. The distribution of the eigenvalues of a matrix having finite probability p of nonzero Gaussian distributed random elements per row follows the Wigner semicircular law in the limit $p\to 1$. For very small p also, which corresponds to the sparse random matrix, one gets a semicircular law with several peaks at different eigenvalues [22].

With the increasing availability of large maps of real-world networks, analyses of spectral densities of adjacency matrix of real-world networks and model networks having real-world properties have also begun [23–26]. These analyses show that the matrix constructed by *zero* and *one* elements corresponding to a unweighted random network also follows the Wigner semicircular law [23] with degeneracy at $\lambda=0$. Small-world model networks show very complex spectral density with many sharp peaks [26], while the spectral density of the scale-free model networks exhibits a so-called triangular distribution [23,25,26]. Spectral density and NNSD of the random matrices constructed by *zero* and *one* elements have been studied extensively in Ref. [27]. These studies show that NNSD of the random matrices follow the GOE distribution of RMT.

III. RANDOM MATRIX STATISTICS

In the random matrix studies of eigenvalues spectra, one has to consider two kinds of properties: (i) global properties, like spectral density or distribution of eigenvalues $\rho(\lambda)$, and (ii) local properties, like eigenvalue fluctuations around $\rho(\lambda)$. Among these, the eigenvalue fluctuations is the most popular one. This is generally obtained from the NNSD of the eigenvalues. The eigenvalues of the network are denoted by λ_i, $i=1,\ldots,N$, where N is the size of the network and $\lambda_i<\lambda_{i+1}$, $\forall i$. In order to get universal properties of the eigenvalue fluctuations, one has to remove the spurious effects due to the variations of spectral density and to work at constant spectral density on the average. Thereby, it is customary in RMT to unfold the eigenvalues by a transformation $\bar{\lambda}_i=\bar{N}(\lambda_i)$, where $\bar{N}(\lambda)=\int_{\lambda_{\min}}^{\lambda}\rho(\lambda')d\lambda'$ is the averaged integrated eigenvalue density [1]. Since the analytical form for $\bar{N}$ is not known, we numerically unfold the spectrum by polynomial curve fitting. Using the unfolded spectrum, we calculate the nearest-neighbor spacings as

$$s_1^{(i)}=\bar{\lambda}_{i+1}-\bar{\lambda}_i,$$

and due to the above unfolding, the average nearest-neighbor spacings $\langle s_1\rangle$ becomes *unity*, being independent of the system. The NNSD $P(s_1)$ is defined as the probability distribution of these $s_1^{(i)}$'s. In the case of Poisson statistics,

$$P(s_1)=\exp(-s_1), \tag{1}$$

whereas for GOE

$$P(s_1)=\frac{\pi}{2}s_1\exp\left(-\frac{\pi s_1^2}{4}\right). \tag{2}$$

For the intermediate cases, NNSD is described by the Brody formula [28]:

$$P_\beta(s_1)=As_1^\beta\exp(-Bs_1^{\beta+1}), \tag{3a}$$

where A and B are determined by the parameter β as follows:

$$A=(1+\beta)\alpha \quad \text{and} \quad \alpha=\left[\Gamma\left(\frac{\beta+2}{\beta+1}\right)\right]^{\beta+1}. \tag{3b}$$

This is a semiempirical formula characterized by the single parameter β, popularly known as the Brody parameter. $\beta=1$ corresponds to the GOE statistics and $\beta=0$ corresponds to the Poisson statistics.

Apart from NNSD, the *next*-nearest-neighbor spacings distribution (NNNSD) is also used to characterize the statistics of eigenvalues fluctuations. We calculate this distribution $P(s_2)$ of next-nearest-neighbor spacings,

$$s_2^{(i)}=(\bar{\lambda}_{i+2}-\bar{\lambda}_i)/2, \tag{4}$$

between the unfolded eigenvalues. A factor of 2 in the denominator is inserted to make the average of next-nearest-neighbor spacings $\langle s_2 \rangle$ *unity*. According to Ref. [1], the NNNSD of GOE matrices is identical to the NNSD of Gaussian symplectic ensemble (GSE) matrices, i.e.,

$$P(s_2) = \frac{2^{18}}{3^6 \pi^3} s_2^4 \exp\left(-\frac{64}{9\pi} s_2^2\right). \tag{5}$$

The NNSD and NNNSD reflect only local correlations among the eigenvalues. The spectral rigidity, measured by the Δ_3 statistic of RMT, gives information about the long-range correlations among eigenvalues and is a more sensitive test for RMT properties of the matrix under investigation [1,29]. In the following we describe the procedure to calculate this quantity.

The Δ_3 statistic measures the least-square deviation of the spectral staircase function representing the averaged integrated eigenvalue density $\bar{N}(\lambda)$ from the best straight line fitting for a finite interval L of the spectrum, i.e.,

$$\Delta_3(L;x) = \frac{1}{L} \min_{a,b} \int_x^{x+L} [N(\bar{\lambda}) - a\bar{\lambda} - b]^2 d\bar{\lambda}, \tag{6}$$

where a and b are obtained from a least-square fit. The average over several choices of x gives the spectral rigidity $\Delta_3(L)$. For the Poisson case, when the eigenvalues are uncorrelated, $\Delta_3(L)=L/15$, reflecting strong fluctuations around the spectral density $\rho(\lambda)$. On the other hand, for the GOE case, $\Delta_3(L)$ depends *logarithmically* on L, i.e.,

$$\Delta_3(L) \sim \frac{1}{\pi^2} \ln L. \tag{7}$$

IV. RESULTS

In the following we present results for the ensemble averaged NNSD, NNNSD, and Δ_3 statistic of random, scale-free, and small-world networks.

A. Random network

First we consider an ensemble of random networks generated by using the Erdös-Rényi algorithm. Starting with $N=2000$ nodes, random connections between pairs of nodes are made with probability p. The average degree of the graph is $k=2n/N=p(N-1)\sim pN$. There exists a critical probability $p_c(N)$ for which one gets a large connected component. The degree distribution of this random graph is a binomial distribution $P(k)=C_{N-1}^k p^k(1-p)^{N-1-k}$. For $p=0.01$, this method yields a connected network with average degree $p\times N=20$. Note that for a very small value of p one gets several disconnected components. In this study the choice of p is high enough to give a connected component typically spanning all the nodes.

We calculate the eigenvalues spectrum of the network generated according to the above algorithm. First the eigenvalues are unfolded by using the technique described in Sec. III. This method yields the eigenvalues with constant spectral density on the average. These unfolded eigenvalues are used to calculate NNSD. The same procedure is repeated for an ensemble of the networks generated for different random realizations. Note that p is always chosen such that the algorithm generates a network with average degree $k=20$. Figure 1(a) plots the ensemble average of NNSD. By fitting this ensemble averaged NNSD with the Brody formula given in Eq. (3a) we get an estimation of the Brody parameter $\beta=0.9786\sim 1$. This value of the Brody parameter clearly indicates the GOE behavior of the NNSD [Eq. (2)]. The inset of Fig. 1(a) shows the corresponding spectral density which follows the well known Wigner's semicircular distribution. The same unfolded eigenvalues are used to calculate NNNSD. For this we calculate next-nearest-neighbor spacings as given in Eq. (4) and plot their distribution in Fig. 2(a). It can be seen from the figure that the NNNSD agrees well with the NNSD of GSE matrices as given in Eq. (5).

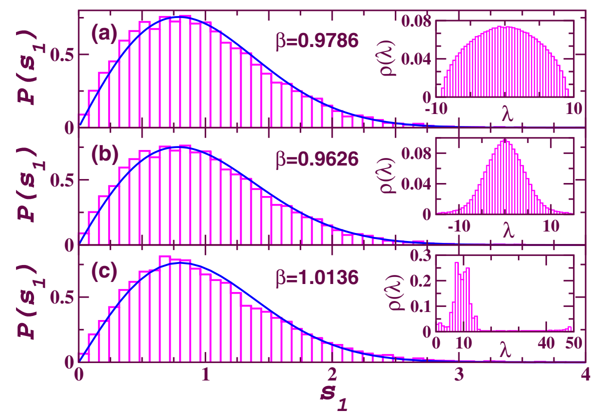

FIG. 1. (Color online) Nearest-neighbor spacings distribution (NNSD) $P(s_1)$ of the adjacency matrices of different networks [(a) random network, (b) scale-free network, and (c) small-world network]. All follow GOE statistics. The histograms are numerical results and the solid lines represent the fitted Brody distribution [Eq. (3a)]. All networks have $N=2000$ nodes and an average degree $k=20$ per node. Figures are plotted for the average over ten random realizations of the networks. Insets show respective spectral densities $\rho(\lambda)$.

As explained in the Introduction, NNSD and NNNSD only tell about the short range correlations among the eigenvalues. Therefore to probe for the long range correlations we study the $\Delta_3(L)$ statistic of the spectrum of this network. $\Delta_3(L)$ is calculated following Eq. (6). Figure 3 plots this statistic for the same ensemble as used for the NNSD and NNNSD calculations above. It can be seen that the $\Delta_3(L)$ statistic agrees very well with the RMT prediction, given by Eq. (7), up to a very large value of L, i.e., $L\sim 300$. The inset of this figure shows the same in semilogarithmic scale. Here one can see the expected linear behavior of $\Delta_3(L)$ with slope of 0.0978 which is very close to the RMT predicted value $1/\pi^2\sim 0.1013$ [Eq. (7)].

Note that here an ensemble of ten networks of dimension $N=2000$ is considered. Statistical properties of eigenvalue spectra of members of this ensemble have very small deviations from each other and hence justify ensemble averaging calculations [30]. Each individual network in the ensemble follows random matrix predictions with very good accuracy, however, to make the statistical analysis more credible, we

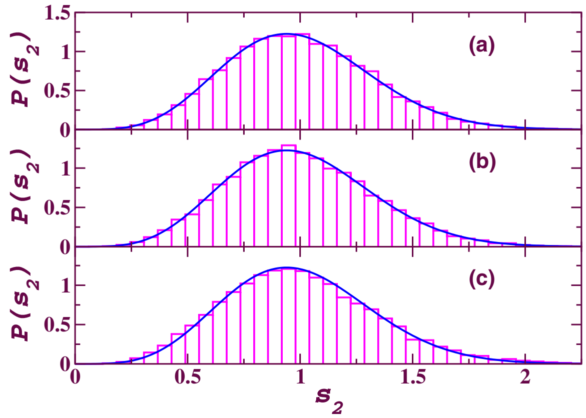

FIG. 2. (Color online) Next-nearest-neighbor spacings distribution (NNNSD) $P(s_2)$ of the adjacency matrices of different networks [(a) random network, (b) scale-free network, and (c) small-world network] is compared with the nearest-neighbor spacings distribution (NNSD) of GSE matrices. Figures are plotted for average over ten realization of the networks. All networks have $N=2000$ nodes and an average degree $k=20$ per node.

present the results for an ensemble of ten networks. Here we would like to mention that an ensemble of networks of much smaller dimensions, say $N=100$, has been studied as well and it follows GOE predictions of RMT. However, for this case, many more realizations are required to get good accuracy.

B. Scale-free network

A scale-free network is generated by using the model of Barabási and Albert [16]. Starting with a small number, m_0 of the nodes, a new node with $m \leq m_0$ connections is added at each time step. This new node connects with an already

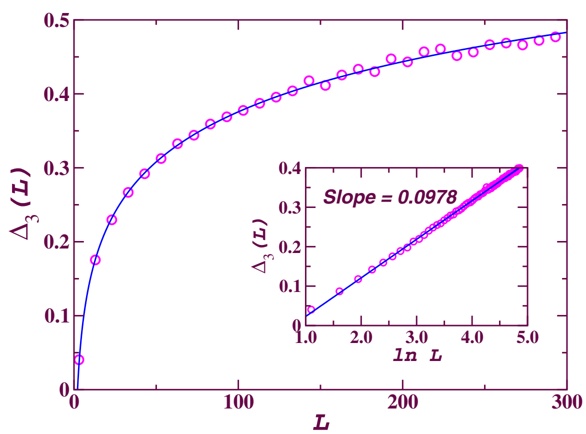

FIG. 3. (Color online) The $\Delta_3(L)$ statistic for eigenvalues spectra of the random network. The circles are numerical results and the solid curve is GOE prediction of RMT. The inset shows the $\Delta_3(L)$ in semilogarithmic scale; in this scale it has the slope 0.0978. The figure is plotted for an average over ten realizations of the networks. All networks have $N=2000$ nodes and an average degree $k=20$ per node.

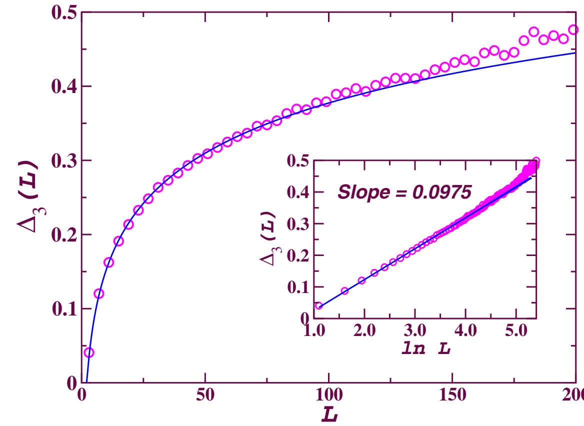

FIG. 4. (Color online) $\Delta_3(L)$ statistic for eigenvalues spectra of the scale-free network. The circles are numerical results and the solid curve is the GOE prediction of RMT. The inset plots the $\Delta_3(L)$ in semilogarithmic scale; in this scale it has the slope 0.0975. The figure is plotted for the average over ten realizations of the networks. All networks have $N=2000$ nodes and an average degree $k=20$ per node.

existing node i with probability $\pi(k_i) \propto k_i$, where k_i is the degree of the node i. After τ time steps the model leads to a network with $N=\tau+m_0$ nodes and $m\tau$ connections. This model generates a scale-free network, i.e., the probability $P(k)$, that a node has degree k and decays as a power law $P(k) \sim k^{-\lambda}$, where λ is a constant and for the type of probability law $\pi(k)$ used here $\lambda=3$. Other forms for the probability $\pi(k)$ are also possible which give different values of λ. However, the results reported here are independent of the value of λ.

Using the above algorithm an ensemble of scale-free networks of size $N=2000$ and average degree $k=20$ is generated. To calculate NNSD, NNNSD, and Δ_3 for the spectra of this ensemble, we follow the same procedure as described in the previous section. Figure 1(b) shows that the NNSD of the scale-free network follows GOE with $\beta=0.9626 \sim 1$. The inset of this figure plots the spectral density of this network. Figure 2(b) shows that the NNNSD of the adjacency matrix of this network agrees well with the NNSD of the GSE matrices. Figure 4 shows the $\Delta_3(L)$ statistic for the adjacency matrix of scale-free network. Here we see that the $\Delta_3(L)$ statistic for the scale-free network agrees very well with the RMT prediction for very large L, i.e., $L \sim 150$, and deviations are seen only after $L=150$. The inset of this figure shows the expected linear behavior of $\Delta_3(L)$ in semilogarithmic scale for $L \lesssim 150$ with the slope of 0.0975, a value very close to the RMT predicted value $1/\pi^2$.

Universality of NNSD and NNNSD for random and scale-free networks seems to give the impression that these networks have the same amount of randomness, but Δ_3 results tell us that the scale-free network is not as random as the random network. This is obvious from their construction algorithms as well, but Δ_3 statistics is capturing this property which is a very important result. The finding also suggests that scale-free networks have some specific features that cannot be modeled by RMT. It may be noted that one can gen-

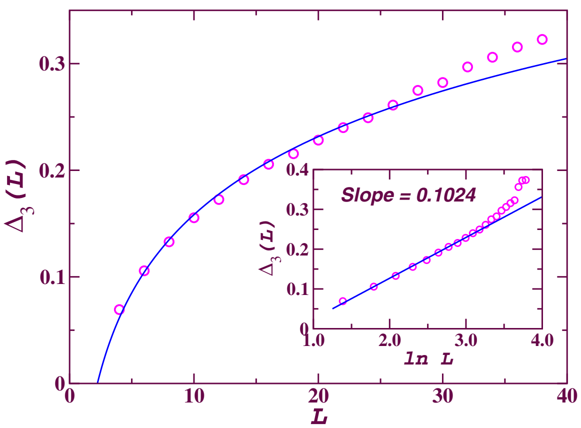

FIG. 5. (Color online) $\Delta_3(L)$ statistic for the eigenvalues spectra of the small-world network. The circles are numerical results and the solid curve is GOE prediction of RMT. The inset plots the $\Delta_3(L)$ up to $L=30$ in the semilogarithmic scale, in this scale $\Delta_3(L)$ has the slope 0.1024. The figure is plotted for the average over ten realizations of the networks. All networks have $N=2000$ nodes and an average degree $k=20$ per node.

erate scale-free networks by using other algorithms as well [31,32], for these networks also spacing distributions and spectral rigidity results will have qualitatively similar behaviors, except that the range of agreement of L with the random matrix prediction would depend upon the amount of randomness in the networks.

C. Small-world network

Small-world networks are constructed using the following algorithm of Watts and Strogatz [15]. Starting with a one-dimensional ring lattice of N nodes in which every node is connected to its $k/2$ nearest neighbors, we randomly rewire each connection of the lattice with the probability p such that self-connections and multiple connections are excluded. Thus $p=0$ gives a regular network and $p=1$ gives a completely random network. The typical small-world behavior is observed around $p=0.005$ [8]. For $N=2000$ and an average degree $k=20$, an ensemble of *ten* different realizations of the network is generated.

Again the same procedure as described in Sec. IV A has been used to calculate NNSD, NNNSD, and Δ_3 for the spectra. Figure 1(c) shows that the NNSD of this network again follows GOE statistics with β very close to 1, i.e., $\beta=1.0136$. The inset shows that the corresponding spectral density is complicated with several peaks. One peak is always at $\lambda=0$. The exact positions of other peaks may vary but the overall form of spectral density remains similar. Figure 2(c) plots the NNNSD of the adjacency matrix of the small-world network. It can be seen that the NNNSD agrees well with the NNSD of GSE matrices. Figure 5 shows the $\Delta_3(L)$ statistic for the spectrum of adjacency matrix corresponding to the small-world network with $p=0.005$. The inset of this figure shows the expected linear behavior of $\Delta_3(L)$ in a semilogarithmic scale for $L\lesssim 30$ with a slope of 0.1024, a value very close to the RMT predicted value $1/\pi^2$. It can be seen here that $\Delta_3(L)$ statistics for the small-world network agrees very well with the RMT prediction for sufficiently large L, i.e., $L\sim 30$, but much less than the same for random and scale-free networks, implying that besides randomness, the small-world network has specific features also. This again suggests that the behavior of Δ_3 statistics can be used to understand the amount of randomness in the networks. More specifically, deviation from the GOE predicted behavior corresponds to the system specific features in the network.

Note that in this paper, results for networks with the average degree 20 are presented. We have studied sparser ($\langle k\rangle<20$) and denser networks ($\langle k\rangle>20$ to $\langle k\rangle\sim N$) as well. The same universal behavior is found for these networks as far as there exists a certain amount of randomness, i.e., the presence of some *minimal* random connections among the nodes. There exist problems with very sparse networks as of average degree 2 and very dense networks as of degree $\sim N$. For sparse networks ($\langle k\rangle\sim 2$) sometimes one can get several degeneracies in eigenvalues [22]. In this case, one has to first get rid of the degeneracies to conclude anything under the RMT framework [33].

Similarly, for dense networks, universal spacing distribution is observed till a very large value of average degree. For $\langle k\rangle\sim N$, the largest eigenvalue has a very high value compared to the rest of $N-1$ eigenvalues which are very close to each other, becoming equal in the limiting case of globally connected networks $\langle k\rangle=N-1$. For example, random networks with $p=0.95$ (which means that the network has 95% of maximum possible connections) also follow RMT predictions of universal spacing distributions till a very large scale. As the number of connections are increased further one starts getting degenerate eigenvalues and for $p=0.999$ high degeneracies at various values (such as $\lambda=-1,0,1$) are observed keeping it trivially out from the RMT studies.

V. DISCUSSIONS

We use RMT to study complex networks and show that in spite of spectral densities of the adjacency matrices being different for different networks, their eigenvalue fluctuations are the same and follow the GOE statistics of RMT. We attribute this universality to the existence of a *minimal amount of randomness* in all these networks and show that randomness in the network connections can be quantified by the Brody parameter. In addition to the NNSD, we present the results of NNNSD and spectral rigidity via the Δ_3 statistic of RMT. The NNNSD of the eigenvalues of these model networks are identical to the NNSD of GSE matrices which again agrees with the RMT prediction given in Eq. (5). NNSD and NNNSD suggest that there exists short range correlations among the eigenvalues. A spectral rigidity test shows that the $\Delta_3(L)$ statistics follows random matrix predicted linear behavior in a semilogarithmic plot for sufficiently large scale L with slope being $\sim 1/\pi^2$ [see Eq. (7)], suggesting long-range correlations among the eigenvalues. The above findings show that statistics of the bulk of eigenvalues of the model networks is consistent with those of a real symmetric random matrix and deviation from this could be understood as a system dependent part.

Universal GOE behavior of NNSD and NNNSD tell us that the networks are sufficiently random, or there exists a *minimal amount of randomness* required to introduce the correlations among the neighboring eigenvalues. The Δ_3 analysis seems to characterize the level of randomness in networks depending on the range of correlations among eigenvalues. Δ_3 analysis of the random network follows the RMT prediction for a very long range of L, which is not very surprising as the random network follows RMT at each level starting from semicircular density distributions. However, interestingly scale-free and small-world networks also follow RMT for a sufficiently large value of L. Beyond this value of L, deviation in the spectral rigidity is seen, indicating a possible breakdown of the universality. This is quite understandable as a small-world network is highly clustered and a scale-free network also has specific features like hubs, so it is natural that they are not as random as the random network. But it is interesting to realize that Δ_3 statistics rightly captures this information. Moreover, the small-world network is generated exactly at the small-world transition by using the Watts and Strogatz algorithm which yields a network with a very high clustering coefficient and very low number of random connections. The results presented in this paper show that this very small number of random connections makes the network *sufficiently* random to introduce the correlations among the eigenvalues for the sufficiently long range.

According to the many recent studies, randomness in connections is one of the most important and desirable ingredients for the proper functionality or the efficient performance of systems having underlying network structures. For instance, information processing in the brain is considered to be because of random connections among different modular structures [34]. We feel that the role of random connections and behavior and evolution of such systems can be studied better under the RMT framework. Also this RMT approach may be used to detect the connections most responsible to increase the *complexity* of networks. For example, the effect of the oxygen molecule on the biochemical network of a metabolic system is recently studied and is shown to increase the complexity of the system leading to a major transition in the history of life [35].

In summary, we use RMT to analyze spectra of complex networks and show that these networks follow universal GOE statistics. These results tell that the random matrix theory, a very well developed branch of physics, can be applied to the complex networks studies. So far we have only concentrated on the model networks studied vastly in the recent literature, providing a basis to the random matrix analysis of networks. Future investigations would involve studies of real-world networks [36].

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